Cmapf: The Elliptic Case

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February 9, 2004

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1. Introduction

For Geographic purposes, the Earth is usually represented as an ellipse of revolution (the *Terrestrial Ellipsoid*), whose minor axis is directed through the North

Pole and the ends of whose major axis sweeps out the Equator. The length of the semi-major axis is designated a and the semi-minor axis b. The *flattening*, f, is defined as one minus the ratio of the two lengths $f = (a-b)/a$, and the *eccentricity* $\varepsilon = \sqrt{1-(b/a)^2}$. Over time, several values have been used for a, b, f and ε:

Name	Date	a(km)	b(km)	$1/f$	ε
Everest	1830	6377.276	6356.075	300.8	8.1473×10^{-2}
Bessel	1841	6377.397	6356.079	299.16	8.1696×10^{-2}
Airy	1858	6377.563	6356.257	299.33	8.1672×10^{-2}
Clarke	1858	6378.294	6356.619	294.27	8.2371×10^{-2}
Clarke (NAD 1927)	1866	6378.206	6356.584	294.99	8.2271×10^{-2}
Clarke	1880	6378.249	6356.515	293.47	8.2483×10^{-2}
Hayford	1909	6378.388	6356.912	297.0	8.1992×10^{-2}
Krassovsky	1948	6378.245	6356.863	298.3	8.1813×10^{-2}
I.U.G.G	1967	6378.160	6356.775	298.25	8.1820×10^{-2}
WGS 84 (NAD 84)	1984	6378.137	6356.7523142	298.25722	8.1819×10^{-2}

2. Location on the Terrestrial Spheroid

A point on the Terrestrial spheroid at a specific latitude θ in the $x-z$ plane is located at a point $(x(\theta), 0, z(\theta))$ on the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, where the tangent vector $\left(\frac{z}{b^2}, -\frac{x}{a^2}\right)$ is proportional to $(\sin(\theta), -\cos(\theta))$, i.e.

$$\frac{z}{x} = \left(\frac{b}{a}\right)^2 \frac{\sin(\theta)}{\cos(\theta)} = \left(1-\varepsilon^2\right)\tan(\theta)$$

then

$$\begin{aligned} a^2 &= x^2 + a^2\frac{z^2}{b^2} \\ &= x^2\left(1 + \left(\frac{b}{a}\right)^2 \tan^2(\theta)\right) \\ &= x^2\left(1 + \left(1-\varepsilon^2\right)\tan^2(\theta)\right) \end{aligned}$$

or

$$\begin{aligned} x &= a\,\frac{\cos(\theta)}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}} \\ z &= a\,\frac{\sin(\theta)\ \ (1-\varepsilon^2)}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}} \end{aligned} \tag{2.1}$$

Since the Terrestrial Spheroid is an ellipsoid of revolution, in the more general case of a longitude λ other than zero, the location in space of a point on the Earth's surface at latitude θ, longitude λ is given by

$$\begin{aligned} x &= r(\theta)\cos(\lambda) \\ y &= r(\theta)\sin(\lambda) \\ z &= a\,\frac{\sin(\theta)\ \ (1-\varepsilon^2)}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}} \\ &= r(\theta)\tan(\theta)\left(1-\epsilon^2\right) \end{aligned} \tag{2.2}$$

where

$$r(\theta) = a\,\frac{\cos(\theta)}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}}$$

2.1. Distance on the Spheroid

The distance on the spheroid between two points whose latitude-longitude coordinates are (θ, λ) and $(\theta + d\theta, \lambda + d\lambda)$ is given by

$$ds^2 = r^2 d\lambda^2 + \left(\left(\frac{dr}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2\right) d\theta^2.$$

Now,

$$\begin{aligned} \frac{dr}{r} &= -\frac{\sin(\theta)}{\cos(\theta)}d\theta + \frac{\varepsilon^2\sin(\theta)\cos(\theta)}{1-\varepsilon^2\sin^2(\theta)}d\theta \\ &= -\frac{\sin(\theta)}{\cos(\theta)}d\theta\ \left(\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\right) \\ dr &= -a\sin(\theta)d\theta\ \frac{1-\varepsilon^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{3}{2}}} \end{aligned}$$

Similarly,

$$
\begin{aligned}
\frac{dz}{z} &= \frac{\cos(\theta)}{\sin(\theta)}d\theta \; + \frac{\varepsilon^2 \sin(\theta)\cos(\theta)}{1-\varepsilon^2\sin^2(\theta)}d\theta \\
&= \frac{\cos(\theta)}{\sin(\theta)}d\theta \; \frac{1}{1-\varepsilon^2\sin^2(\theta)} \\
dz &= a\cos(\theta)d\theta \; \frac{1-\varepsilon^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{3}{2}}}
\end{aligned}
$$

so that

$$
ds^2 = a^2\left(\frac{\cos^2(\theta)}{1-\varepsilon^2\sin^2\theta}d\lambda^2 + \frac{\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2\sin^2\theta\right)^3}\; d\theta^2\right) \tag{2.3}
$$

2.2. Curvature of the Ellipsoid

Extrinsic curvature is the second order rate of change of position, normal to the surface, with respect to distance along the surface. In the North-South direction, we have

$$
\frac{d\mathbf{x}}{ds} = \frac{\partial \mathbf{x}}{\partial\theta}\frac{d\theta}{ds} = \frac{\left|\frac{\partial \mathbf{x}}{\partial\theta}\right|}{\left|\frac{ds}{d\theta}\right|}\mathbf{t}
$$

where $\mathbf{t}$ is a unit vector, clearly tangent to the ellipsoid, and thus normal to $\mathbf{n}$, the normal vector to the surface. The second derivative is

$$
\frac{d^2\mathbf{x}}{ds^2} = \frac{\left|\frac{\partial \mathbf{x}}{\partial\theta}\right|}{\left(\frac{ds}{d\theta}\right)^2}\frac{\partial \mathbf{t}}{\partial\theta} + \mathbf{t}\frac{d}{ds}\left(\frac{\left|\frac{\partial \mathbf{x}}{\partial\theta}\right|}{\left|\frac{ds}{d\theta}\right|}\right)
$$

and, in the direction of the normal,

$$
\mathbf{n}\cdot\frac{d^2\mathbf{x}}{ds^2} = \frac{\left|\frac{\partial \mathbf{x}}{\partial\theta}\right|}{\left(\frac{ds}{d\theta}\right)^2}\mathbf{n}\cdot\frac{\partial \mathbf{t}}{\partial\theta}
$$

Now, $\left|\frac{\partial \mathbf{x}}{\partial\theta}\right| = \frac{ds}{d\theta}$, while by definition of latitude, $\frac{\partial \mathbf{t}}{\partial\theta} = -\mathbf{n}$, so the curvature in this direction is

$$
\mathbf{n}\cdot\frac{d^2\mathbf{x}}{ds^2} = \frac{1}{ds/d\theta} = \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\varepsilon^2\right)}
$$

Similarly, in the East-West direction,

$$\begin{aligned}
\mathbf{n}\cdot\frac{d^2\mathbf{x}}{ds^2} &= \frac{\mathbf{n}\cdot\frac{\partial\mathbf{t}}{\partial\lambda}}{ds/d\lambda} = \frac{\cos\theta}{a\cos\theta/\left(1-\varepsilon^2\sin^2\theta\right)^{1/2}} \\
&= \frac{\left(1-\varepsilon^2\sin^2\theta\right)^{1/2}}{a}
\end{aligned}$$

In an arbitrary direction,

$$\begin{aligned}
\frac{d\mathbf{x}}{ds} &= \frac{\partial\mathbf{x}}{\partial\theta}\frac{d\theta}{ds} + \frac{\partial\mathbf{x}}{\partial\lambda}\frac{d\lambda}{ds} \\
&= \mathbf{b}_n\left|\frac{\partial\mathbf{x}}{\partial\theta}\right|\frac{d\theta}{ds} + \mathbf{b}_e\left|\frac{\partial\mathbf{x}}{\partial\lambda}\right|\frac{d\lambda}{ds}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{n}\cdot\frac{d^2\mathbf{x}}{ds^2} &= \left|\frac{\partial\mathbf{x}}{\partial\theta}\right|\frac{d\theta}{ds}\mathbf{n}\cdot\frac{\partial\mathbf{b}_n}{\partial\theta}\frac{d\theta}{ds} + \left|\frac{\partial\mathbf{x}}{\partial\lambda}\right|\frac{d\lambda}{ds}\mathbf{n}\cdot\frac{\partial\mathbf{b}_e}{\partial\lambda}\frac{d\lambda}{ds} \\
&= -\left|\frac{\partial\mathbf{x}}{\partial\theta}\right|\left(\frac{d\theta}{ds}\right)^2 - \cos\theta\left|\frac{\partial\mathbf{x}}{\partial\lambda}\right|\left(\frac{d\lambda}{ds}\right)^2
\end{aligned}$$

3. The Mercator Projection

A conformal projection on a cylinder yields a Mercator projection. If the radius of the cylinder is a, and the equator maps into the cross-section at $z = 0$, then the latitude longitude point (θ, λ) maps into a point

$$\begin{aligned}
x &= a\lambda \\
y &= am(\theta)
\end{aligned} \tag{3.1}$$

where $m(\theta)$ is a function that must satisfy the conformal conditions that the map factor in the $N - S$ direction must be the same as in the $E - W$ direction at any point; i.e.

$$\frac{am'(\theta)}{\partial s/\partial\theta} = \frac{a}{\partial s/\partial\lambda}$$

or

$$m'(\theta) = \frac{\left(1-\varepsilon^2\right)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)}$$

which integrates to

$$
\begin{aligned}
m(\theta) &= \int \frac{(1-\varepsilon^2)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)} d\theta \\
&= \int \frac{(1-\varepsilon^2)}{\left(1-\sin^2\theta\right)\left(1-\varepsilon^2\sin^2(\theta)\right)} d\sin\theta \\
&= \int \left[\frac{1}{2(1-\sin\theta)} + \frac{1}{2(1+\sin\theta)} - \frac{1}{2}\frac{\varepsilon^2}{1-\varepsilon\sin\theta} - \frac{1}{2}\frac{\varepsilon^2}{1+\varepsilon\sin\theta}\right] d\sin\theta \\
&= \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) - \frac{\varepsilon}{2}\ln\left(\frac{1+\varepsilon\sin\theta}{1-\varepsilon\sin\theta}\right)
\end{aligned} \tag{3.2}
$$

3.1. Inverting the Mercator Projection

Equation (3.1) (3.2) allows the transition from latitude-longitude (λ, θ) to the Mercator projection's (x, y) coordinates. A transformation in the other direction, finding $\theta(m)$ from m does not lend itself to a closed form solution. One appropriate solution is to obtain a first estimate θ_0 solving

$$\frac{1}{2}\ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right) = m$$

in the form

$$\sin\theta_0 = \frac{\exp(m) - \exp(-m)}{\exp(m) + \exp(-m)}$$

and then recursively solve

$$\frac{1}{2}\ln\left(\frac{1+\sin\theta_k}{1-\sin\theta_k}\right) = m + \frac{\varepsilon}{2}\ln\left(\frac{1+\varepsilon\sin\theta_{k-1}}{1-\varepsilon\sin\theta_{k-1}}\right)$$

for $\sin\theta_k$, $k = 1, \dots$.

Alternatively, we could use the Newton-Raphson formulation

$$\theta_{k+1} = \theta_k + \frac{m - m(\theta_k)}{m'(\theta_k)}$$

where

$$m'(\theta_k) = \frac{(1-\varepsilon^2)}{\cos(\theta_k)\left(1-\varepsilon^2\sin^2(\theta_k)\right)}$$

and

$$m(\theta_k) = \frac{1}{2}\ln\left(\left(\frac{1+\sin\theta_k}{1-\sin\theta_k}\right)\left(\frac{1-\varepsilon\sin\theta_k}{1+\varepsilon\sin\theta_k}\right)^{\epsilon}\right)$$

3.2. Scale of the Mercator Projection

The map factor of the Mercator projection from the world to the Mercator projection μ_{wm} is the ratio of distance on the projection to that on the Terrestrial ellipse; i.e.

$$\mu_{wm}(\theta) = \frac{a d\lambda}{r(\theta) d\lambda} = \frac{\left(1 - \varepsilon^2 \sin^2(\theta)\right)^{\frac{1}{2}}}{\cos(\theta)}$$

3.3. Curvature (Scale Gradient)

The Curvature vector is the logarithmic gradient of the scale and is a vector in the direction of North, with magnitude

$$\begin{aligned}
G &= \frac{\partial}{\partial\theta}\left(\ln \mu_{wm}(\theta)\right) / \left(\partial s / \partial\theta\right) \\
&= \frac{\left(1 - \epsilon^2 \sin^2(\theta)\right)^{3/2}}{a\left(1 - \epsilon^2\right)} \frac{\partial}{\partial\theta}\left(\ln \mu_{wm}(\theta)\right) \\
&= \frac{\left(1 - \epsilon^2 \sin^2(\theta)\right)^{3/2}}{a\left(1 - \epsilon^2\right)} \left(\frac{\sin(\theta)}{\cos(\theta)} - \frac{\varepsilon^2 \sin(\theta)\cos(\theta)}{\left(1 - \varepsilon^2 \sin^2(\theta)\right)}\right) \\
&= \frac{\left(1 - \epsilon^2 \sin^2(\theta)\right)^{3/2}}{a\left(1 - \epsilon^2\right)} \left(\frac{\sin(\theta)\left(1 - \varepsilon^2 \sin^2(\theta)\right) - \varepsilon^2 \sin(\theta)\cos^2(\theta)}{\cos(\theta)\left(1 - \varepsilon^2 \sin^2(\theta)\right)}\right) \\
&= \frac{\sin(\theta)}{a\cos(\theta)} \left(1 - \epsilon^2 \sin^2(\theta)\right)^{1/2}
\end{aligned}$$

4. The Lambert Conformal Projection

The Lambert Conformal projection is computed from the Mercator projection by specifying a *tangent latitude* ϕ_t and calculating the power $\gamma = \sin\phi_t$. In canonical coordinates ξ, η, we have

$$\begin{aligned}
\xi &= \exp(-\gamma m) \frac{\sin(\gamma(\lambda - \lambda_0))}{\gamma} \\
\eta &= \frac{1 - \exp(-\gamma m)\cos(\gamma(\lambda - \lambda_0))}{\gamma}
\end{aligned}$$

so that the origin of the ξ, η system is at the equator $y_m m = 0$, longitude $\lambda = \lambda_0$.

The inverse of the transformation is performed by first converting to the Mercator Projection, and thence as in the previous section, to the Earth. The first stage is done as follows:

$$\begin{aligned}
\lambda &= \lambda_0 + \tan^{-1}\left(\frac{\gamma\xi}{1-\gamma\eta}\right) \\
m &= -\frac{1}{2\gamma}\ln\left(1-\gamma\left(2\eta+\gamma\left(\xi^2+\eta^2\right)\right)\right)
\end{aligned}$$

4.1. Lambert Conformal scale

The map factor of the transition from Mercator map to Lambert Conformal is

$$\mu_{ml} = \frac{d\xi^2 + d\eta^2}{dy_m^2 + y_m^2 d\lambda^2} = \exp\left(-\gamma m\right)$$

and combined with the map factor from world to Mercator,

$$\begin{aligned}
\mu_{wl} &= \mu_{wm}\mu_{ml} = \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}}{\cos(\theta)}\exp\left(-\gamma m\right) \\
&= \left(\frac{\left(1-\varepsilon\sin\theta\right)^{1-\gamma\epsilon}\left(1+\varepsilon\sin\theta\right)^{1+\gamma\epsilon}}{\left(1+\sin\theta\right)^{1+\gamma}\left(1-\sin\theta\right)^{1-\gamma}}\right)^{1/2}
\end{aligned}$$

The logarithmic derivative of this map factor with respect to latitude θ follows

$$\begin{aligned}
\frac{1}{\mu}\frac{d\mu}{d\theta} &= \frac{1}{2}\left[-\frac{(1-\gamma\epsilon)\,\varepsilon\cos\theta}{1-\varepsilon\sin\theta} + \frac{(1+\gamma\epsilon)\,\varepsilon\cos\theta}{1+\varepsilon\sin\theta} - \frac{(1+\gamma)\cos\theta}{1+\sin\theta} + \frac{(1-\gamma)\cos\theta}{1-\sin\theta}\right] \\
&= \varepsilon^2\cos\theta\frac{\gamma-\sin\theta}{(1-\varepsilon\sin\theta)(1+\varepsilon\sin\theta)} - 2\cos\theta\frac{\gamma-\sin\theta}{(1+\sin\theta)(1-\sin\theta)} \\
&= -\cos\theta\,(\gamma-\sin\theta)\left(\frac{1}{\left(1-\sin^2\theta\right)} - \frac{\varepsilon^2}{\left(1-\varepsilon^2\sin^2\theta\right)}\right) \\
&= -\frac{\cos\theta\,(\gamma-\sin\theta)}{\left(1-\sin^2\theta\right)}\frac{\left(1-\varepsilon^2\right)}{\left(1-\varepsilon^2\sin^2\theta\right)} \\
&\quad -\frac{(\gamma-\sin\theta)}{\cos\theta}\frac{\left(1-\varepsilon^2\right)}{\left(1-\varepsilon^2\sin^2\theta\right)}
\end{aligned}$$

and demonstrates that the maximum scale occurs where $\sin\theta = \gamma$, i.e. at the tangent latitude.

4.2. EQVLAT

To find the tangent latitude (i.e. γ) at which the scale for two separate latitudes θ_1 and θ_2 are equal, we must solve for γ

$$\begin{aligned} \mu_{wl}(\theta_1) &= \mu_{wl}(\theta_2) \\ \mu_{wm}(\theta_1)\exp(-\gamma m_1) &= \mu_{wm}(\theta_2)\exp(-\gamma m_2) \end{aligned}$$

whence γ is found by

$$\gamma = \frac{\ln\left(\mu_{wm}(\theta_2)/\mu_{wm}(\theta_1)\right)}{(m_2 - m_1)}$$

or

$$\gamma = \frac{2\ln\left(\frac{\cos(\theta_1)}{\cos(\theta_2)}\right) - \ln\left(\frac{1-\varepsilon^2\sin^2(\theta_1)}{1-\varepsilon^2\sin^2(\theta_2)}\right)}{2(m_2 - m_1)}$$

or

$$\gamma = \frac{\ln\left(\frac{1-\sin\theta_1}{1-\epsilon\sin\theta_1}\right) - \ln\left(\frac{1-\sin\theta_2}{1-\epsilon\sin\theta_2}\right) + \ln\left(\frac{1+\sin\theta_1}{1+\epsilon\sin\theta_1}\right) - \ln\left(\frac{1+\sin\theta_2}{1+\epsilon\sin\theta_2}\right)}{2(m_2 - m_1)}$$

or

$$\gamma = \frac{\ln\left(\frac{1-\sin^2(\theta_1)}{1-\varepsilon^2\sin^2(\theta_1)} \Big/ \frac{1-\sin^2(\theta_2)}{1-\varepsilon^2\sin^2(\theta_2)}\right)}{2(m_2 - m_1)}$$

Using the above formulae to find γ, we then find the tangent latitude as $\theta_t = \sin^{-1}\gamma$.

4.3. Curvature (Scale Gradient)

The Curvature vector is the logarithmic gradient of the scale and is a vector in the direction of North, with magnitude

$$
\begin{aligned}
G &= \frac{\partial}{\partial\theta}\left(\ln\left(\mu_{wm}(\theta)\,\mu_{ml}(\theta)\right)\right)/\left(\partial s/\partial\theta\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\left(\frac{\partial}{\partial\theta}\left(\ln\left(\mu_{wm}(\theta)\right)\right)+\frac{\partial m}{\partial\theta}\frac{\partial}{\partial m}\left(\ln\left(\mu_{ml}(m)\right)\right)\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\left(\frac{\sin(\theta)\left(1-\varepsilon^2\right)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)}-\gamma\frac{\partial m}{\partial\theta}\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\left(\frac{\sin(\theta)\left(1-\varepsilon^2\right)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)}-\gamma\frac{\left(1-\varepsilon^2\right)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)}\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{1/2}}{a\cos(\theta)}\left(\sin(\theta)-\gamma\right) \\
\mu_{wm}\mu_{ml} &= \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}}{\cos(\theta)}\exp\left(-\gamma m\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\frac{\partial}{\partial\theta}\left(\ln\mu_{wm}(\theta)\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\left(\frac{\sin(\theta)}{\cos(\theta)}-\frac{\varepsilon^2\sin(\theta)\cos(\theta)}{\left(1-\varepsilon^2\sin^2(\theta)\right)}\right) \\
&= \frac{\left(1-\epsilon^2\sin^2(\theta)\right)^{3/2}}{a\left(1-\epsilon^2\right)}\left(\frac{\sin(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)-\varepsilon^2\sin(\theta)\cos^2(\theta)}{\cos(\theta)\left(1-\varepsilon^2\sin^2(\theta)\right)}\right) \\
&= \frac{\sin(\theta)}{a\cos(\theta)}\left(1-\epsilon^2\sin^2(\theta)\right)^{1/2}
\end{aligned}
$$

A. The Conformal Sphere

The conformal sphere is a sphere on which points are mapped from the Clark Spheroid in such a way that distance measures on the sphere are conformal to those on the spheroid. That is, the scale of the map at any point in the North-South direction is the same as the scale in the East-West direction at the same point.

Suppose the point (θ, λ) on the Clark spheroid is mapped to the point (ϑ, λ) on the conformal sphere, where the "latitude" ϑ is known as the "Conformal Latitude". The ratio of the distances on the conformal sphere to those on the spheroid in the East-West direction is

$$\mu = \frac{\cos(\vartheta)}{a\cos(\theta)}\left(1 - \varepsilon^2 \sin^2(\theta)\right)^{\frac{1}{2}}$$

while the same ratio in the North-South direction is

$$\mu = \frac{d\vartheta}{a\ d\theta}\ \frac{\left(1 - \varepsilon^2 \sin^2(\theta)\right)^{\frac{3}{2}}}{(1 - \varepsilon^2)}.$$

If these two expressions for μ are to be equal, we must have

$$\frac{d\vartheta}{\cos\vartheta} = \frac{d\theta}{\cos(\theta)}\ \frac{1 - \varepsilon^2}{1 - \varepsilon^2 \sin^2(\theta)}$$

Solution by partial fractions:

$$\frac{d\sin\vartheta}{2}\left(\frac{1}{1 - \sin\vartheta} + \frac{1}{1 + \sin\vartheta}\right) = \frac{d\sin\theta}{2}\left(\frac{1}{1 - \sin\theta} + \frac{1}{1 + \sin\theta} - \frac{\varepsilon^2}{1 - \varepsilon\sin(\theta)} - \frac{\varepsilon^2}{1 + \varepsilon\sin(\theta)}\right)$$

so

$$y = \frac{1}{2}\ \ln\left(\frac{1 + \sin\vartheta}{1 - \sin\vartheta}\right) = \frac{1}{2}\ \ln\left(\frac{1 + \sin\theta}{1 - \sin\theta}\right) - \frac{\varepsilon}{2}\ \ln\left(\frac{1 + \varepsilon\sin(\theta)}{1 - \varepsilon\sin(\theta)}\right).$$

The function $y(\theta)$ defined here is the *Mercator index* and represents the y-coordinate of a Mercator projection for which the x-coordinate is the longitude in radians. The similar expression

$$\widehat{y}(\theta) = \frac{1}{2}\ \ln\left(\frac{1 + \sin\theta}{1 - \sin\theta}\right)$$

represents the *spherical Mercator index*, and is the y-coordinate of what would be a Mercator map if the Earth were a sphere.

The mapfactor $\mu(\theta)$ of the transformation from the Clark spheroid to the conformal sphere is given by

$$\mu_S = \frac{\cos(\vartheta)}{a\cos(\theta)}\left(1 - \varepsilon^2 \sin^2(\theta)\right)^{\frac{1}{2}}$$

or, substituting for $\cos(\vartheta)$ from

$$
\begin{aligned}
\sin\vartheta &= \frac{\exp(y)-\exp(-y)}{\exp(y)+\exp(-y)} \\
\cos\vartheta &= \frac{2}{\exp(y)+\exp(-y)}
\end{aligned}
$$

we have

$$\mu_S = \frac{2}{a}\,\frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1+\varepsilon}{2}}}{(1+\sin(\theta))(1-\varepsilon\sin(\theta))^{\varepsilon}+(1-\sin(\theta))(1+\varepsilon\sin(\theta))^{\varepsilon}}$$

The map factor for the transformation from the Clark Spheroid to the Mercator projection is the combination of the transformation from Clark Spheroid to the conformal sphere with the transformation from the conformal sphere to the Mercator projection and is given by

$$\mu_M = \frac{\mu_S}{\cos\vartheta} = \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{\frac{1}{2}}}{a\cos(\theta)}$$

Finding the Mercator index, given a latitude, is straight-forward. The inverse process of finding latitude, given a Mercator index, requires first evaluating the spherical Mercator index. We accomplish this by creating a sequence of such indices. Let $\widehat{y}_0 = y$, then let

$$\exp(\widehat{y}_{n+1}) = \exp(y)\ \left(\frac{(1+\varepsilon)\exp(\widehat{y}_n)+(1-\varepsilon)\exp(-\widehat{y}_n)}{(1-\varepsilon)\exp(\widehat{y}_n)+(1+\varepsilon)\exp(-\widehat{y}_n)}\right)^{\frac{\varepsilon}{2}}$$

If the $\widehat{y}_n$ tend to a limit $\widehat{y}_\infty$ as $n \to \infty$, that limit will satisfy ?. Then

$$\sin(\theta) = \frac{\exp(\widehat{y}_\infty)-\exp(-\widehat{y}_\infty)}{\exp(\widehat{y}_\infty)+\exp(-\widehat{y}_\infty)}$$

and

$$\cos(\theta) = \frac{2}{\exp(\widehat{y}_\infty)+\exp(-\widehat{y}_\infty)}$$

B. Scratch Pad

Projection from a point (x_0, y_0, z_0) to a point on the ellipsoid. Ideally, the projection should be "vertical", along a path that intersects the geoid at a right angle. We conceive the geoid as a member of a family of confocal ellipsoids, i.e. ellipsoids of revolution whose ellipses share the same foci. A family of hyperbolae, also sharing the same foci.

Without loss of generality, we assume $y = y_0 = 0$, and look for a solution to the system

$$\begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{a^2\left(1-e^2\right)} &= 1 \\ \frac{x^2}{m^2} - \frac{z^2}{a^2e^2-m^2} &= 1 \\ \frac{x_0^2}{m^2} - \frac{z_0^2}{a^2e^2-m^2} &= 1 \end{aligned}$$

For amy $m < ae$, the second equation represents an hyperbola confocal to the ellipse, and hence orthogonal to it. The third equation can be solved for the value of m representing the particular ellipse going through the given point.

We solve the first two equations for x^2:

$$\begin{aligned} \left(1-e^2\right)x^2 + z^2 &= a^2\left(1-e^2\right) \\ \left(\frac{a^2e^2}{m^2}-1\right)x^2 - z^2 &= a^2e^2-m^2 \end{aligned}$$

or

$$\begin{aligned} x^2 &= \frac{a^2\left(1-e^2\right)+a^2e^2-m^2}{\left(1-e^2\right)+\left(\frac{a^2e^2}{m^2}-1\right)} \\ &= \frac{m^2}{e^2} \end{aligned}$$

whence

$$z^2 = \left(a^2-x^2\right)\left(1-e^2\right)$$

Now, from the third equation,

$$\left(a^2e^2-m^2\right)x_0^2 - m^2z_0^2 = m^2\left(a^2e^2-m^2\right)$$

$$m^4 - m^2\left(x_0^2 + z_0^2 - a^2e^2\right) + a^2e^2x_0^2 = 0$$

or

$$x^2 = \frac{m^2}{e^2} = \frac{1}{2e^2}\left[\left(x_0^2 + z_0^2 - a^2e^2\right) - \left(\left(x_0^2 + z_0^2 - a^2e^2\right)^2 - 4x_0^2a^2e^2\right)^{1/2}\right]$$

the minus sign being chosen in order to ensure $m < ae$. Note that $\left(x_0^2 + z_0^2 - a^2e^2\right)^{1/2}$ represents the distance from the point (x_0, z_0) to the nearest focus $(ae, 0)$.

In the limit as $e \to 0$,

$$x^2 = \frac{m^2}{e^2} \to \frac{x_0^2}{x_0^2 + z_0^2}a^2$$

which represents projection along the radius to the sphere.

C. Scratchpad 2 Geodesics

Total distance between point 1: $x_i(t_0)$ and point 2: $x_i(t_1)$ is

$$\int_{t_0}^{t_1}\left(g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)^{1/2}dt$$

where $x_i(t)$ is a function of t from t_0 to t_1. If we condider a series of curves, parameterized by σ, $x_i(t;\sigma)$, where $x_i(t_0;\sigma)$ and $x_i(t_1;\sigma)$ are fixed values for all σ, we look for an extremum in total distance that satisfies

$$\frac{\delta}{\delta\sigma}\int_{t_0}^{t_1}\left(g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)^{1/2}dt = 0$$

or

$$\int_{t_0}^{t_1}\frac{1}{2g}\left(\frac{\partial g_{ij}}{\partial x_k}\frac{dx_k}{d\sigma}\frac{dx_i}{dt}\frac{dx_j}{dt} + 2g_{ij}\frac{dx_i}{dt}\frac{d^2x_j}{dtd\sigma}\right)dt = 0$$

where $g = \left(g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)^{1/2}$. After integration by parts,

$$\int_{t_0}^{t_1}\frac{1}{2g}\left(\frac{\partial g_{ij}}{\partial x_k}\frac{dx_k}{d\sigma}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)dt - \int_{t_0}^{t_1}\frac{dx_j}{d\sigma}\frac{d}{dt}\left(\frac{1}{g}g_{ij}\frac{dx_i}{dt}\right)dt = 0$$

$$\int_{t_0}^{t_1}\frac{1}{2g}\left(\frac{\partial g_{ij}}{\partial x_k}\frac{dx_k}{d\sigma}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)dt - \int_{t_0}^{t_1}\frac{1}{g}\frac{dx_j}{d\sigma}\left(\frac{\partial g_{ij}}{\partial x_k}\frac{dx_k}{dt}\frac{dx_i}{dt} + gg_{ij}\frac{d}{dt}\left(\frac{1}{g}\frac{dx_i}{dt}\right)\right)dt = 0$$

$$\int_{t_0}^{t_1} \frac{1}{2g}\frac{dx_k}{d\sigma}\left(\frac{\partial g_{ij}}{\partial x_k}\frac{dx_i}{dt}\frac{dx_j}{dt} - 2\frac{\partial g_{ik}}{\partial x_j}\frac{dx_j}{dt}\frac{dx_i}{dt} - 2gg_{ik}\frac{d}{dt}\left(\frac{1}{g}\frac{dx_i}{dt}\right)\right)dt = 0$$

Allowing $dx_k/d\sigma$ to vary over all possible ranges, we have the equation

$$\left(\frac{\partial g_{ij}}{\partial x_k} - 2\frac{\partial g_{ik}}{\partial x_j}\right)\frac{dx_j}{dt}\frac{dx_i}{dt} - 2g_{ik}g\frac{d}{dt}\left(\frac{1}{g}\frac{dx_i}{dt}\right) = 0$$

We define the Christoffel symbol of the first kind, $[ij,k]$ by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right)$$

and the Christoffel symbol of the second kind by

$$\Gamma^k_{ij} = g^{kl}\,[ij,l]$$

Then we have

$$\frac{d}{dt}\left(\frac{1}{g}\frac{dx_i}{dt}\right) + \frac{1}{g}\Gamma^i_{jk}\frac{dx_j}{dt}\frac{dx_k}{dt} = 0$$

On the spheroid, if $x_1 = \theta$ is latitude and $x_2 = \lambda$ is longitude, then

$$g_{ij} = a^2\begin{bmatrix} \frac{\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^3} & 0 \\ 0 & \frac{\cos^2(\theta)}{1-\varepsilon^2\sin^2(\theta)} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial g_{11}}{\partial x_1} &= 6a^2\varepsilon^2\frac{\sin\theta\cos\theta\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^4} \\ \frac{\partial g_{22}}{\partial x_1} &= -2a^2\frac{\left(1-\varepsilon^2\right)\cos\theta\sin\theta}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2} \end{aligned}$$

and

$$[ij,1] = a^2\begin{bmatrix} 3\varepsilon^2\frac{\sin\theta\cos\theta\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^4} & 0 \\ 0 & \frac{\left(1-\varepsilon^2\right)\cos\theta\sin\theta}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2} \end{bmatrix}$$

while

$$[ij,2] = a^2 \begin{bmatrix} 0 & -\frac{\left(1-\varepsilon^2\right)\cos\theta\sin\theta}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2} \\ -\frac{\left(1-\varepsilon^2\right)\cos\theta\sin\theta}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2} & 0 \end{bmatrix}$$

On expansion,

$$\Gamma^1_{ij} = \begin{bmatrix} \frac{3\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\sin\theta\cos\theta & 0 \\ 0 & \frac{1-\varepsilon^2\sin^2(\theta)}{1-\varepsilon^2}\cos\theta\sin\theta \end{bmatrix}$$

while

$$\Gamma^2_{ij} = \begin{bmatrix} 0 & -\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\frac{\sin\theta}{\cos\theta} \\ -\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\frac{\sin\theta}{\cos\theta} & 0 \end{bmatrix}$$

In standard coordinate form,

$$\begin{aligned} \frac{d^2\theta}{ds^2} &= \frac{-3\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\sin\theta\cos\theta\left(\frac{d\theta}{ds}\right)^2 - \frac{1-\varepsilon^2\sin^2(\theta)}{1-\varepsilon^2}\cos\theta\sin\theta\left(\frac{d\lambda}{ds}\right)^2 \\ \frac{d^2\lambda}{ds^2} &= 2\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\frac{\sin\theta}{\cos\theta}\frac{d\theta}{ds}\frac{d\lambda}{ds} \end{aligned}$$

where we have used $g = ds/dt$ to convert from t to s as independent variable. From the second equation,

$$\begin{aligned} \frac{d}{ds}\ln\left(\frac{d\lambda}{ds}\right) &= 2\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\frac{\sin\theta}{\cos\theta}\frac{d\theta}{ds} \\ &= \frac{d}{ds}\ln\left(\frac{1-\varepsilon^2\sin^2(\theta)}{\cos^2(\theta)}\right) \end{aligned}$$

$\int 2\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2(\theta)}\frac{\sin\theta}{\cos\theta}d\theta = \int 2\frac{1-\varepsilon^2}{1-\varepsilon^2x^2}\frac{xdx}{(1-x^2)} = \int\frac{1-\varepsilon^2}{1-\varepsilon^2t}\frac{dt}{(1-t)} = \ln\left(-1+\varepsilon^2t\right) - \ln\left(-1+t\right)$
or

$$\frac{d\lambda}{ds} = K\left(\frac{1-\varepsilon^2\sin^2(\theta)}{\cos^2(\theta)}\right)$$

where K is an arbitrary constant, determined by the initial conditions of the geodesic.

By definition of the metric,

$$\begin{aligned} 1 &= g_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} \\ &= g_{11}\left(\frac{d\theta}{ds}\right)^2 + g_{22}\left(\frac{d\lambda}{ds}\right)^2 \end{aligned}$$

or

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1 - g_{22}\left(\frac{d\lambda}{ds}\right)^2}{g_{11}}$$

Now,

$$g_{22}\left(\frac{d\lambda}{ds}\right)^2 = K^2\frac{1-\varepsilon^2\sin^2(\theta)}{\cos^2(\theta)}$$

so

$$\begin{aligned}
\left(\frac{d\theta}{ds}\right)^2 &= \frac{1 - K^2\frac{1-\varepsilon^2\sin^2(\theta)}{\cos^2(\theta)}}{\frac{(1-\varepsilon^2)^2}{\left(1-\varepsilon^2\sin^2(\theta)\right)^3}} \\
&= \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^4}{(1-\varepsilon^2)^2}\left(\frac{1}{1-\varepsilon^2\sin^2(\theta)} - \frac{K^2}{\cos^2(\theta)}\right) \\
&= \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^3}{(1-\varepsilon^2)^2\cos^2(\theta)}\left(\left(1-K^2\right) - \left(1-\varepsilon^2K^2\right)\sin^2(\theta)\right)
\end{aligned}$$

$$\frac{d\theta}{ds} = \pm\frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^{3/2}}{(1-\varepsilon^2)}\left(1-\left(1-\varepsilon^2\sin^2(\theta)\right)\left(\frac{K}{\cos(\theta)}\right)^2\right)^{1/2}$$

This means that the geodesic will oscillate between $\theta_{\max}$ and $\theta_{\min}$ which have the values of the solutions of

$$\sin^2(\theta) = \frac{(1-K^2)}{(1-\varepsilon^2K^2)}$$

Further,

$$
\begin{aligned}
\frac{d^2\theta}{ds^2} &= \frac{d}{ds}\left(\frac{d\theta}{ds}\right) \\
&= \frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{d\theta}{ds}\right) \\
&= \frac{1}{2}\frac{d}{d\theta}\left(\frac{d\theta}{ds}\right)^2 \\
&= \frac{1}{2}\frac{d}{d\theta}\left(\frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^4}{\left(1-\varepsilon^2\right)^2}\left(\frac{1}{1-\varepsilon^2\sin^2(\theta)}-\frac{K^2}{\cos^2(\theta)}\right)\right) \\
&= \frac{1}{2\left(1-\varepsilon^2\right)^2}\left[-4\left(1-\varepsilon^2\sin^2(\theta)\right)^3\left(2\varepsilon^2\sin(\theta)\cos(\theta)\right)\left(\frac{1}{1-\varepsilon^2\sin^2(\theta)}-\frac{K^2}{1-\sin^2(\theta)}\right)\right] \\
&\quad + \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^4}{2\left(1-\varepsilon^2\right)^2}\left(\frac{2\varepsilon^2\sin(\theta)\cos(\theta)}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2}-\frac{2K^2\sin(\theta)\cos(\theta)}{\left(1-\sin^2(\theta)\right)^2}\right) \\
&= \frac{2\sin(\theta)\cos(\theta)}{2\left(1-\varepsilon^2\right)^2}\left[-4\varepsilon^2\left(1-\varepsilon^2\sin^2(\theta)\right)^3\left(\frac{1-K^2-\left(1-K^2\varepsilon^2\right)\sin^2(\theta)}{\left(1-\varepsilon^2\sin^2(\theta)\right)\left(1-\sin^2(\theta)\right)}\right)\right] \\
&\quad + \frac{\left(1-\varepsilon^2\sin^2(\theta)\right)^4}{2\left(1-\varepsilon^2\right)^2}\left(2\varepsilon^2\sin(\theta)\cos(\theta)\right)\left(\frac{1}{\left(1-\varepsilon^2\sin^2(\theta)\right)^2}-\frac{K^2}{\left(1-\sin^2(\theta)\right)^2}\right)
\end{aligned}
$$

D. Scratchpad 3 Geodesics

Drawing a curve from point 1 $(\theta_1, \lambda_1) = (\theta(t_1), \lambda(t_1))$ to point 2 $(\theta_2, \lambda_2) = (\theta(t_2), \lambda(t_2))$, the total distance along the curve is

$$S = \int_{t_1}^{t_2}\left\{g^2(\theta)\left[\frac{d\theta}{dt}\right]^2 + h^2(\theta)\left[\frac{d\lambda}{dt}\right]^2\right\}^{1/2} dt$$

If S is a minimum, corresponding to $(\theta(t), \lambda(t))$, then a shift to the curve $(\theta(t)+\delta\theta(t), \lambda(t)+\delta\lambda(t))$, where $\delta\theta$ and $\delta\lambda$ are arbitrary, vanish at $t = t_1$ and $t = t_2$, and "small", results in a δS which should vanish

$$\delta S = \int_{t_1}^{t_2} \frac{gg'\left[d\theta/dt\right]^2 d\delta\theta/dt + g^2\left[d\theta/dt\right] d\delta\theta/dt + hh'\left[d\lambda/dt\right]^2 d\delta\theta/dt + h^2\left[d\lambda/dt\right] d\delta\lambda/dt}{\left\{g^2(\theta)\left[d\theta/dt\right]^2 + h^2(\theta)\left[d\lambda/dt\right]^2\right\}^{1/2}} dt = 0$$

Letting $\delta\theta = 0$, and integrating by parts, we have

$$\int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{h^2\,[d\lambda/dt]}{\left\{g^2\,(\theta)\,[d\theta/dt]^2 + h^2\,(\theta)\,[d\lambda/dt]^2\right\}^{1/2}}\right]\delta\lambda\,(t)\,dt = 0$$

for all $\delta\lambda$, whence

$$\frac{d}{dt}\left[\frac{h^2\,[d\lambda/dt]}{\left\{g^2\,(\theta)\,[d\theta/dt]^2 + h^2\,(\theta)\,[d\lambda/dt]^2\right\}^{1/2}}\right] = 0$$

so that

$$h^2\frac{d\lambda}{dt} = M\left\{g^2\,(\theta)\,[d\theta/dt]^2 + h^2\,(\theta)\,[d\lambda/dt]^2\right\}^{1/2} = M\frac{ds}{dt}$$

for a constant M. Using arc length s instead of the parameter t, we have

$$h^2\frac{d\lambda}{ds} = M$$

and we find for θ,

$$g\frac{d\theta}{ds} = \pm\left(1 - \left(\frac{M}{h}\right)^2\right)^{1/2}$$

$$\frac{d}{ds}\left(g\frac{d\theta}{ds}\right) = \pm\left(\frac{h^2}{h^2 - M^2}\right)^{1/2}\frac{M^2}{h^3}\frac{dh}{d\theta}\frac{d\theta}{ds} = \frac{M^2}{gh^3}\frac{dh}{d\theta}$$

so long as $h \geq M$. Writing

$$1 = g^2\left[\frac{d\theta}{ds}\right]^2 + h^2\left[\frac{d\lambda}{ds}\right]^2 = g^2\left[\frac{d\theta}{ds}\right]^2 + \left[\frac{M}{h}\right]^2$$

and differentiating w.r.t. s, we have

$$\begin{aligned} g^2\frac{d^2\theta}{ds^2} &= \frac{M^2}{h^3}h' - gg'\left[\frac{d\theta}{ds}\right]^2 \\ \frac{d^2\theta}{ds^2} &= \frac{M^2}{g^2h^3}h' - \frac{g'}{g}\left[\frac{d\theta}{ds}\right]^2 \end{aligned}$$

$$\begin{aligned} g^2\,(\theta) &= \frac{\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2\sin^2\theta\right)^3} \\ h^2\,(\theta) &= \frac{\cos^2(\theta)}{1-\varepsilon^2\sin^2\theta} \end{aligned}$$

E. Minimizing Integrals

To minimize

$$I = \int_a^b L\left(\theta, \dot{\theta}, \lambda, \dot{\lambda}\right) dt$$

we seek $\theta(t), \lambda(t)$ so that I is a minimum, so that $\delta I = 0$ where $I + \delta I$ is obtained from $\bar{\theta} + \delta\theta, \lambda + \delta\lambda$. For small $\delta\theta, \delta\lambda$, we have

$$\delta I = \int_a^b \left(L_\theta \delta\theta + L_{\dot{\theta}} \delta\dot{\theta} + L_\lambda \delta\lambda + L_{\dot{\lambda}} \delta\dot{\lambda}\right) dt = 0$$

or, after integration by parts,

$$\delta I = \int_a^b \left(L_\theta - \frac{d}{dt} L_{\dot{\theta}}\right) \delta\theta + \left(L_\lambda - \frac{d}{dt} L_{\dot{\lambda}}\right) \delta\lambda dt = 0$$

and, since this should hold for all $\delta\theta$ and $\delta\lambda$ which vanish at $t = a$ and $t = b$, we have the Lagrangian equations

$$\begin{aligned} \frac{d}{dt} L_{\dot{\theta}} &= L_\theta \\ \frac{d}{dt} L_{\dot{\lambda}} &= L_\lambda \end{aligned}$$

Applying the above to

$$L\left(\theta, \dot{\theta}, \lambda, \dot{\lambda}\right) = \left\{g^2(\theta)\,\dot{\theta}^2 + h^2(\theta)\,\dot{\lambda}^2\right\}^{1/2}$$

we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{g^2\dot{\theta}}{L}\right) &= \frac{gg'\dot{\theta}^2 + hh'\dot{\lambda}^2}{L} \\ \frac{d}{dt}\left(\frac{h^2\dot{\lambda}}{L}\right) &= 0 \end{aligned}$$

If we re-parametrize so that $ds = Ldt$, then

$$g^2(\theta)\left(\frac{d\theta}{ds}\right)^2 + h^2(\theta)\left(\frac{d\lambda}{ds}\right)^2 = 1$$

$$h^2 \frac{d\lambda}{ds} = M = \text{const.}$$

and

$$\begin{aligned}
\frac{d}{ds}\left(g^2 \frac{d\theta}{ds}\right) &= gg'\left(\frac{d\theta}{ds}\right)^2 + hh'\left(\frac{d\lambda}{ds}\right)^2 \\
&= \frac{g'}{g^3}\left(g^2 \frac{d\theta}{ds}\right)^2 + \frac{h'}{h^3} M^2 \\
\frac{d^2\theta}{ds^2} &= -gg'\left(\frac{d\theta}{ds}\right)^2 + hh'\left(\frac{d\lambda}{ds}\right)^2 \\
&= -\frac{g'}{g} + \left(\frac{g'h^2}{g} + hh'\right)\left(\frac{d\lambda}{ds}\right)^2 \\
&= -\frac{g'}{g} + \left(\frac{g'}{g} + \frac{h'}{h}\right)\frac{M^2}{h^2} \\
&= -\frac{g'}{g}\left(1 - \frac{M^2}{h^2}\right) + \frac{h'}{h}\frac{M^2}{h^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\left(g^2 \frac{d\theta}{ds}\right) &= gg'\left(\frac{d\theta}{ds}\right)^2 + hh'\left(\frac{d\lambda}{ds}\right)^2 \\
&= \frac{g'}{g^3}\left(g^2 \frac{d\theta}{ds}\right)^2 + \frac{h'}{h^3} M^2 \\
\frac{d}{ds}\left(g \frac{d\theta}{ds}\right) &= hh'\left(\frac{d\lambda}{ds}\right)^2 \\
&= \frac{h'}{h}\frac{M^2}{h^2}
\end{aligned}$$

From

$$\begin{aligned}
g^2(\theta) &= \frac{\left(1-\varepsilon^2\right)^2}{\left(1-\varepsilon^2 \sin^2\theta\right)^3} \\
h^2(\theta) &= \frac{\cos^2(\theta)}{1-\varepsilon^2 \sin^2\theta}
\end{aligned}$$

we have

$$\begin{aligned}
\frac{g'}{g} &= 3\frac{\varepsilon^2 \sin\theta\cos\theta}{1-\varepsilon^2\sin^2\theta} = \frac{\sin\theta}{\cos\theta}\frac{3\varepsilon^2\cos^2\theta}{1-\varepsilon^2\sin^2\theta} \\
\frac{h'}{h} &= -\frac{\sin\theta}{\cos\theta} + \frac{\varepsilon^2\sin\theta\cos\theta}{1-\varepsilon^2\sin^2\theta} \\
&= -\frac{\sin\theta}{\cos\theta}\left(\frac{1-\varepsilon^2}{1-\varepsilon^2\sin^2\theta}\right)
\end{aligned}$$

F. Bibliography

References

[Abramowitz and Stegun] , 1964, National Bureau of Standards, Washington, DC,